Fourth Memoir on a New Theory of Symmetric Functions.

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§15.

194. At the conclusion of the previous memoir I applied the linear operations

$$g_0,\ g_1,\ g_{-1},\ \ldots\ldots$$

directly to the theory of separations; I proceed to the further development of this part of the subject; it will be merely necessary in general to consider symmetric functions symbolized by positive non-zero integers.

A previous result, given in §14, may be written

$$\frac{(-)^s}{s} g_s = \sum_{\pi} \sum_{p} \frac{(-)^{\Sigma\pi}(\Sigma\pi - 1)!}{\ldots\ldots \pi_3!\,\pi_2!\,\pi_1!} (\ldots\ldots 3^{p_3} 2^{p_2} 1^{p_1})\, \partial_{(\ldots\ldots 3^{p_3+\pi_3} 2^{p_2+\pi_2} 1^{p_1+\pi_1})};$$

the right-hand side may be broken up into fragments, in each of which the numbers

$$\ldots\ldots \pi_3,\ \pi_2,\ \pi_1$$

are constant.

We may thus write

$$\frac{(-)^s g_s}{s} = \sum_{\pi} \frac{(-)^{\Sigma\pi}(\Sigma\pi - 1)!}{\ldots\ldots \pi_3!\,\pi_2!\,\pi_1!} \sum_{p} (\ldots\ldots 3^{p_3} 2^{p_2} 1^{p_1})\, \partial_{(\ldots\ldots 3^{p_3+\pi_3} 2^{p_2+\pi_2} 1^{p_1+\pi_1})},$$

where the linear operator

$$\sum_{p} (\ldots\ldots 3^{p_3} 2^{p_2} 1^{p_1})\, \partial_{(\ldots\ldots 3^{p_3+\pi_3} 2^{p_2+\pi_2} 1^{p_1+\pi_1})},$$

in which the numbers

$$\ldots\ldots \pi_3,\ \pi_2,\ \pi_1$$

are constant, and the summation is merely in regard to the numbers

$$\ldots\ldots p_3,\ p_2,\ p_1,$$

that is, to every separate

$$(\ldots\ldots 3^{p_3+\pi_3}2^{p_2+\pi_2}1^{p_1+\pi_1}),$$

is one of the fragments above mentioned.

This fragmentary operation has of course a weight s; but further it may be regarded as having a partition

$$(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})$$

of the number s; we may so define the fragment and may write, for brevity and convenience,

$$\sum_p (\ldots\ldots 3^{p_3}2^{p_2}1^{p_1})\,\partial_{(\ldots\ldots 3^{p_3+\pi_3}2^{p_2+\pi_2}1^{p_1+\pi_1})} = g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}.$$

The operator relation is now written

$$\frac{(-)^s g_s}{s} = \sum_\pi \frac{(-)^{\Sigma\pi}(\Sigma\pi-1)!}{\ldots\ldots \pi_3!\,\pi_2!\,\pi_1!} g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})},$$

the summation being in regard to every partition

$$(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1}),$$

of the number s, which occurs in the given separable partition.

195. Considering a perfectly general separable partition, every partition of s may occur, and it is, in consequence, convenient to discuss the full result

$$\frac{(-)^s}{s} g_s = \sum \frac{(-)^{\Sigma\pi}(\Sigma\pi-1)!}{\ldots\ldots \pi_3!\,\pi_2!\,\pi_1!} g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})},$$

the summation having reference to every partition of the number s.

196. Just as in the ordinary theory we meet with a linear differential operation corresponding to every number, so here in the wider theory of separations we are brought face to face with a linear differential operation which is in correspondence with an arbitrary partition of an arbitrary number.

197. In the simplest cases we have the equivalences

$$\begin{aligned}
g_1 &= g_{(1)}, \\
g_2 &= g_{(1^2)} - 2g_{(2)}, \\
g_3 &= g_{(1^3)} - 3g_{(21)} + 3g_{(3)}, \\
g_4 &= g_{(1^4)} - 4g_{(21^2)} + 2g_{(2^2)} + 4g_{(31)} - 4g_{(4)}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

each *weight operator* $g_1, g_2, g_3, g_4, \ldots\ldots$ being expressible as a linear function of its fragmentary *partition operators* according to the same law as the sums of powers are represented by elementary (that is, unitary) symmetric functions.

198. Let
$$g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})},\quad g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})},$$
be any two partition operators of the same or of different weights. We have the known theorem:

$$g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\,g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} = \overline{g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\,g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})}} + g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})} \dagger g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})},$$

where the bar written over the product on the right denotes that the operators are to be multiplied together symbolically and the symbol $\dagger$ denotes the performance of the operation $g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}$ upon the operator $g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})}$, where the latter is considered to be a function of symbols of quantity only.

Now since

$$g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})} = \sum(\ldots\ldots 3^{j_3}2^{j_2}1^{j_1})\,\partial_{(\ldots\ldots 3^{j_3+\pi_3}2^{j_2+\pi_2}1^{j_1+\pi_1})},$$

$$g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} = \sum(\ldots\ldots 3^{j_3+\pi_3}2^{j_2+\pi_2}1^{j_1+\pi_1})\,\partial_{(\ldots\ldots 3^{j_3+\pi_3+\rho_3}2^{j_2+\pi_2+\rho_2}1^{j_1+\pi_1+\rho_1})},$$

there results

$$g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})} \dagger g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} = g_{(\ldots\ldots 3^{\pi_3+\rho_3}2^{\pi_2+\rho_2}1^{\pi_1+\rho_1})},$$

and, thence, the multiplication theorem

$$g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\,g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} = \overline{g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\,g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})}} + g_{(\ldots\ldots 3^{\pi_3+\rho_3}2^{\pi_2+\rho_2}1^{\pi_1+\rho_1})};$$

analogous to the known theorem

$$g_s g_t = \overline{g_s g_t} + g_{s+t},$$

with which it should be contrasted.

199. We are now led to the result:

$$g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\,g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})} - g_{(\ldots\ldots 3^{\rho_3}2^{\rho_2}1^{\rho_1})}\,g_{(\ldots\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})} = 0.$$

The expression on the left has been termed by Sylvester (in his Lectures on Reciprocants and elsewhere) the alternant of the operators involved.

Theorem. The *alternant* of any two *partition operators* vanishes.

200. This theorem again leads us to two corollaries—

Corollary 1. The *alternant* of any *partition operator* and any weight operator vanishes.

Corollary 2. The *alternant* of any two weight operators vanishes.

This last corollary is already well known.

The theorem may be otherwise stated by enunciating a necessary consequence.

201. *Theorem.* Any partition operator and any weight operator is commutable with any other partition or weight operator.

202. Consider now the solutions of the linear partial differential equation

$$P=0,$$

where P is any partition or weight operator.

If ϕ be one solution, so that identically

$$P\phi=0,$$

it follows that $Q\phi$ must be another solution, where Q is any other partition or weight operator.

For $$PQ\phi-QP\phi=0,$$

and since $$P\phi=0,$$

therefore also $$P(Q\phi)=0,$$

or $Q\phi$ is also a solution of

$$P=0.$$

203. *Theorem.* If ϕ be a solution of $P=0$, $Q\phi$ is also a solution when P and Q are any two partition or weight operations.

204. The partition operators are of most importance in the case of the differential equation

$$g_s=0.$$

For we have seen that

$$\frac{(-)^s g_s}{s}=\sum_\pi \frac{(-)^{\Sigma\pi}(\Sigma\pi-1)!}{\ldots\ \pi_3!\ \pi_2!\ \pi_1!}\,g(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1}),$$

and if $g_s\phi=0$, where ϕ is expressed in terms of separations of the separable partition

$$(\ldots\ 3^{p_3}2^{p_2}1^{p_1}),$$

the effect of the partition operator

$$g(\ldots 3^{\pi_3}2^{\pi_2}1^{\pi_1})$$

will be the production of terms each of which is a separation of the partition

$$(\ldots . \, 3^{P_3-\pi_3}2^{P_2-\pi_2}1^{P_1-\pi_1}).$$

No other partition operator can produce separations of this partition. Since therefore

$$g_s\phi$$

is identically zero, we must have also

$$g_{(\ldots . 3^{\pi_3}2^{\pi_2}1^{\pi_1})}\,\phi = 0\,.$$

205. *Theorem.* If a function be annihilated by a *weight operator*, it must also be annihilated by each *partition operator* of that weight.

This important and comprehensive theorem renders the calculation of tables of separations a straightforward and comparatively easy matter.

206. As an example, suppose we have to calculate the function (3^2) in terms of separations of (31^3).

Assume

$$2(3^2) = A(31^3) + B(31^2)(1) + C(3)(1^3) + D(31)(1^2) + E(31)(1)^2 + F(3)(1^2)(1),$$

there being of necessity no term $(3)(1)^3$.

We have

$$\begin{aligned}
g_{(1)} &= \partial_{(1)} + (1)\,\partial_{(1^2)} + (1^2)\,\partial_{(1^3)} + (3)\,\partial_{(31)} + (31)\,\partial_{(31^2)} + (31^2)\,\partial_{(31^3)},\\
g_{(1^2)} &= \partial_{(1^2)} + (1)\,\partial_{(1^3)} + (3)\,\partial_{(31^2)} + (31)\,\partial_{(31^3)},\\
g_{(1^3)} &= \partial_{(1^3)} + (3)\,\partial_{(31^3)},\\
g_{(3)} &= \partial_{(3)} + (1)\,\partial_{(31)} + (1^2)\,\partial_{(31^2)} + (1^3)\,\partial_{(31^3)},\\
g_{(31)} &= \partial_{(31)} + (1)\,\partial_{(31^2)} + (1^2)\,\partial_{(31^3)},\\
g_{(31^2)} &= \partial_{(31^2)} + (1)\,\partial_{(31^3)},\\
g_{(31^3)} &= \partial_{(31^3)},
\end{aligned}$$

and these are the only operations we are concerned with; their number is $(1+1)(3+1)-1=7$, viz. one for each separate. The weight operators which annihilate the function are g_1, g_2, g_4 and g_5. Hence we have as annihilators the partition operators

$$g_{(1)},\ g_{(1^2)},\ g_{(31)},\ g_{(31^2)}.$$

Further,

$$g_3 2(3^2) = -6(3),$$
$$g_6 2(3^2) = +6,$$

that is,

$$[\partial_{(1^3)} + (3)\,\partial_{(31^3)} + 3\{\partial_{(3)} + (1)\,\partial_{(31)} + (1^2)\,\partial_{(31^2)} + (1^3)\,\partial_{(31^3)}\}]\,2\,(3^2) = -\,6\,(3),$$
$$6\,\partial_{(31^3)}\,2\,(3^2) = +\,6\,.$$

These operations are more than sufficient to determine all the coefficients and to verify the result to be found in Vol. XI, p. 17 of this Journal.

§16.

207. Recalling the equivalences

$$\begin{aligned}
g_1 &= g_{(1)}\,,\\
g_2 &= g_{(1^2)} - 2g_{(2)}\,,\\
g_3 &= g_{(1^3)} - 3g_{(21)} + 3g_{(3)}\,,\\
g_4 &= g_{(1^4)} - 4g_{(21^2)} + 2g_{(2^2)} + 4g_{(31)} - 4g_{(4)}\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

I find it convenient to write them in a new notation, so as to bring into better evidence their law of formation. I write

$$\begin{aligned}
g_{[1]} &= g_{[1]}\,,\\
g_{[2]} &= g_{[1]^2} - 2g_{[1^2]}\,,\\
g_{[3]} &= g_{[1]^3} - 3g_{[1^2][1]} + 3g_{[1^3]}\,,\\
g_{[4]} &= g_{[1]^4} - 4g_{[1^2][1]^2} + 2g_{[1^2]^2} + 4g_{[1^3][1]} - 4g_{[1^4]}\,,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

where $g_{[1^\lambda][1^\mu]\ldots\ldots}$ takes the place of $g_{(\lambda\mu\ldots\ldots)}$ and $g_{[\lambda]}$ that of g_λ. This notation is quite consistent. In general $g_{[\lambda\ldots\ldots][\mu\ldots\ldots]\ldots\ldots}$ denotes that the operator is formed according to the same law as the symmetric function $(\lambda\ldots\ldots)(\mu\ldots\ldots)\ldots\ldots$ when expressed in terms of elementary symmetric functions.

The theorems

$$\begin{aligned}
g_\lambda \dagger g_\mu &= g_{\lambda+\mu}\,,\\
g_{(\lambda\mu\ldots\ldots)} \dagger g_{(\pi\rho\ldots\ldots)} &= g_{(\lambda\mu\ldots\ldots\pi\rho\ldots\ldots)}
\end{aligned}$$

now become in the new notation

$$\begin{aligned}
g_{[\lambda]} \dagger g_{[\mu]} &= g_{[\lambda][\mu]}\,,\\
g_{[1^\lambda][1^\mu]\ldots\ldots} \dagger g_{[1^\pi][1^\rho]\ldots\ldots} &= g_{[1^\lambda][1^\mu]\ldots\ldots[1^\pi][1^\rho]\ldots\ldots}\,,
\end{aligned}$$

that is, a partition suffix addition is transformed into a partition suffix multiplication.

208. We have now in succession

$$
\begin{aligned}
g_{[\lambda]}g_{[\mu]} &= \overline{g_{[\lambda]}g_{[\mu]}} + g_{[\lambda][\mu]}\,,\\
g_{[\kappa]}g_{[\lambda]}g_{[\mu]} &= \overline{g_{[\kappa]}g_{[\lambda]}g_{[\mu]}} + \overline{g_{[\kappa]}g_{[\lambda][\mu]}} + \overline{g_{[\lambda]}g_{[\mu][\kappa]}} + \overline{g_{[\mu]}g_{[\kappa][\lambda]}} + g_{[\kappa][\lambda][\mu]}\,,\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

which are collateral with the symmetric function theorems:

$$
\begin{aligned}
(\lambda)(\mu) &= (\lambda\mu) + (\lambda+\mu)\,,\\
(\varkappa)(\lambda)(\mu) &= (\varkappa\lambda\mu) + (\varkappa,\ \lambda+\mu) + (\lambda,\ \mu+\varkappa) + (\mu,\ \varkappa+\lambda) + (\varkappa+\lambda+\mu)\,,\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

and further,

$$
\begin{aligned}
\overline{g_{[\lambda]}g_{[\mu]}} &= g_{[\lambda]}g_{[\mu]} - g_{[\lambda][\mu]}\,,\\
\overline{g_{[\kappa]}g_{[\lambda]}g_{[\mu]}} &= g_{[\kappa]}g_{[\lambda]}g_{[\mu]} - g_{[\kappa]}g_{[\lambda][\mu]} - g_{[\lambda]}g_{[\mu][\kappa]} - g_{[\mu]}g_{[\kappa][\lambda]} + 2g_{[\kappa][\lambda][\mu]}\,,\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

in correspondence with the known results:

$$
\begin{aligned}
(\lambda\mu) &= s_\lambda s_\mu - s_{\lambda+\mu}\,,\\
(\varkappa\lambda\mu) &= s_\kappa s_\lambda s_\mu - s_\kappa s_{\lambda+\mu} - s_\lambda s_{\mu+\kappa} - s_\mu s_{\kappa+\lambda} + 2s_{\kappa+\lambda+\mu}\,,\\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

209. In both systems of formulae the same modifications, in particular cases, are necessary.

N. B.—Note the result,

$$
g_{[\lambda][\mu]} = g_{[\lambda\mu]} + g_{[\lambda+\mu]}\,,
$$

and so on; similar developments proceed exactly as in the case of symmetric functions.

§17.

210. Reverting again to the previous notation,

$$
\begin{aligned}
g_1 &= g_{(1)}\,,\\
g_2 &= g_{(1^2)} - 2g_{(2)}\,,\\
\text{etc.} &= \text{etc.},
\end{aligned}
$$

and recalling the well-known relations, viz.

$$
\begin{aligned}
g_1 &= G_1\,,\\
g_2 &= G_1^2 - 2G_2\,,\\
g_3 &= G_1^3 - 3\,G_2G_1 + 3\,G_3\,,\\
\text{etc.} &= \text{etc.},
\end{aligned}
$$

where G_1, G_2, G_3 .... are the obliterators before met with, we notice the similarity of the two laws of operation.

211. We deduce

$$\begin{aligned}
G_1 &= g_1 = g_{(1)},\\
2G_2 &= g_1^2 - g_2 = g_{(1)}^2 - g_{(1^2)} + 2g_{(2)},\\
6G_3 &= g_1^3 - 3g_2g_1 + 2g_3,\\
&= g_{(1)}^3 - 3g_{(1^2)}g_{(1)} + 2g_{(1^3)} + 6\{g_{(2)}g_{(1)} - g_{(21)}\} + 6g_{(3)},\\
24G_4 &= g_1^4 - 6g_2g_1^2 + 3g_2^2 + 8g_3g_1 - 6g_4,\\
&= g_{(1)}^4 - 6g_{(1^2)}g_{(1)}^2 + 3g_{(1^2)}^2 + 8g_{(1^3)}g_{(1)} - 6g_{(1^4)}\\
&\quad + 12\{g_{(2)}g_{(1)}^2 - g_{(2)}g_{(1^2)} - 2g_{(21)}g_{(1)} + 2g_{(21^2)}\} + 12\{g_{(2)}^2 - g_{(2^2)}\}\\
&\quad + 24\{g_{(3)}g_{(1)} - g_{(31)}\} + 24g_{(4)},
\end{aligned}$$

after some slight reduction.

212. We can thus always express the obliterators G_1, G_2, G_3, ... in terms of successive operations of the linear partition operators.

To see the law, on the right-hand side of the identity, last obtained, replace each partition by a partition containing a single part equal in magnitude to the weight of the partition and omit the literal symbols altogether. We thus obtain

$$\begin{aligned}
&\{(1)^4 - 6(2)(1)^2 + 3(2)^2 + 8(3)(1) - 6(4)\} + 12\{(2)(1)^2 - (2)^2 - 2(3)(1) + 2(4)\}\\
&\qquad + 12\{(2)^2 - (4)\} + 24\{(3)(1) - (4)\} + 24(4),
\end{aligned}$$

which is

$$\begin{aligned}
&24(1^4) + 12.2(21^2) + 12.2(2^2) + 24(31) + 24(4)\\
&\qquad = 24\{(1^4) + (21^2) + (2^2) + (31) + (4)\}.
\end{aligned}$$

It is easy to establish *à priori* that this must be so. The expression of $24G_4$ breaks up into 5 portions corresponding to the 5 partitions of the number 4. In general the expression for G_s breaks up into as many portions as there are partitions of s, and in general

$$G_s = \sum \frac{(-)^{L_1+L_2+\ldots\ldots+l+m+\ldots\ldots}}{L_1!\,L_2!\ldots\ldots}\left\{\frac{(l_1+m_1+\ldots-1)!}{l_1!\,m_1!\ldots\ldots}\right\}^{L_1}\left\{\frac{(l_2+m_2+\ldots-1)!}{l_2!\,m_2!\ldots\ldots}\right\}^{L_2}$$

$$\ldots\ldots g^{L_1}_{(\lambda_1{}^{l_1}\mu_1{}^{m_1}\ldots\ldots)}g^{L_2}_{(\lambda_2{}^{l_2}\mu_2{}^{m_2}\ldots\ldots)}\ldots\ldots,$$

where $\qquad (\lambda_1^{l_1}\mu_1^{m_1}\ldots\ldots)^{L_1}(\lambda_2^{l_2}\mu_2^{m_2}\ldots\ldots)^{L_2}\ldots\ldots$

is any separation of any partition

$$(\lambda^l\mu^m\ldots\ldots)$$

of the number s.

213. Instead of expressing G_s in terms of successive operations of linear partition operators, we may express it in terms of operators formed by multiplying together the partition operators symbolically. The system of relations is

$$\begin{aligned}
G_1 &= g_{(1)}, \\
2G_2 &= \overline{g^2_{(1)}} + 2g_{(2)}, \\
6G_3 &= \overline{g^3_{(1)}} + 6\overline{g_{(2)}g_{(1)}} + 6g_{(3)}, \\
24G_4 &= \overline{g^4_{(1)}} + 12\overline{g_{(2)}g^2_{(1)}} + 12\overline{g^2_{(2)}} + 24\overline{g_{(3)}g_{(1)}} + 24g_{(4)}, \\
\text{etc.} &= \text{etc.},
\end{aligned}$$

and in general

$$G_s = \sum \frac{1}{l!\,m!\,\ldots\ldots}\,\overline{g^l_{(\lambda)}g^m_{(\mu)}\ldots\ldots}$$

§18.

214. In what has preceded in respect of the linear operations $g_1, g_2, g_3, \ldots\ldots$ a generalization has been made from a number to the partition of a number and weight operators were broken up into linear functions of partition operators.

A like generalization can be made in respect of the obliterating operators $G_1, G_2, G_3, \ldots\ldots$

215. Suppose a symmetric function

$$f(a_1, a_2, a_3, \ldots\ldots a_s, \ldots\ldots) = f$$

to be the product of m monomial functions, and write

$$f = f_1 f_2 f_3 \ldots\ldots f_m.$$

If a_s be changed into $a_s + \mu a_{s-1}$, we have from previous work

$$\begin{aligned}
&(1 + \mu G_1 + \mu^2 G_2 + \mu^3 G_3 + \ldots\ldots + \mu^s G_s + \ldots\ldots) f \\
&\qquad = (1 + \mu G_1 + \mu^2 G_2 + \ldots\ldots + \mu^s G_s + \ldots\ldots) f_1 \\
&\qquad \times (1 + \mu G_1 + \mu^2 G_2 + \ldots\ldots + \mu^s G_s + \ldots\ldots) f_2 \\
&\qquad \times \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\qquad \times (1 + \mu G_1 + \mu^2 G_2 + \ldots\ldots + \mu^s G_s + \ldots\ldots) f_m,
\end{aligned}$$

and now expanding and equating coefficients of like powers of μ, there result:

$$\begin{aligned}
G_1 f &= \Sigma (G_1 f_1) f_2 f_3 \ldots\ldots f_m, \\
G_2 f &= \Sigma (G_1 f_1)(G_1 f_2) f_3 \ldots\ldots f_m + \Sigma (G_2 f_1) f_2 \ldots\ldots f_m, \\
G_3 f_1 &= \Sigma (G_1 f_1)(G_1 f_2)(G_1 f_3) f_4 \ldots\ldots f_m \\
&\quad + \Sigma (G_2 f_1)(G_1 f_2) f_3 \ldots\ldots f_m + \Sigma (G_3 f_1) f_2 f_3 \ldots\ldots f_m,
\end{aligned}$$

and so on, the summations being in regard to the different terms obtained by permutation of the m suffixes of the functions $f_1, f_2, \ldots f_m$.

216. In general, in the expression of G_s there appears a summation corresponding to each partition of the number s.

The summation in correspondence with a partition $(p_1, p_2, \ldots p_s)$ is

$$\Sigma\,(G_{p_1}f_1)(G_{p_2}f_2)\ldots(G_{p_s}f_s)f_{s+1}\ldots f_m.$$

Thus, when performed upon a product of functions, the operator G_s breaks up into as many distinct operations as the weight s possesses partitions.

I denote the operation indicated by the summation

$$\Sigma\,(G_{p_1}f_1)(G_{p_2}f_2)\ldots(G_{p_s}f_s)f_{s+1}\ldots f_m$$

by
$$G_{(p_1 p_2 \ldots p_s)},$$

and speak of it as a partition obliterating operator.

217. We have now the equivalence

$$G_s = \Sigma\, G_{(p_1 p_2 \ldots p_s)},$$

the summation being in regard to every partition of the weight s.

This theorem indicates the method of operating with G_s upon a product of symmetric functions.

In particular,

$$\begin{aligned} G_1 &= G_{(1)}, \\ G_2 &= G_{(1^2)} + G_{(2)}, \\ G_3 &= G_{(1^3)} + G_{(21)} + G_{(3)}, \\ &\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

218. Interesting relations may be established between the partition g and the partition G operators.

From the relation

$$\sum_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} g_{(p_1^{\pi_1}p_2^{\pi_2}\ldots)} = \sum_\pi \frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots} G_{p_1}^{\pi_1} G_{p_2}^{\pi_2}\ldots$$

there arise the relations

$$\begin{aligned} g_{(1)} &= G_1 = G_{(1)}, \\ g_{(1^2)} - 2g_{(2)} &= G_1^2 - 2G_2 = G_{(1)}^2 - 2G_{(1^2)} - 2G_{(2)}, \\ g_{(1^3)} - 3g_{(21)} + 3g_{(3)} &= G_1^3 - 3G_2G_1 + 3G_3 \\ &= G_{(1)}^3 - 3G_{(1^2)}G_{(1)} + 3G_{(1^3)} - 3\{G_{(2)}G_{(1)} - G_{(21)}\} + 3G_{(3)}, \end{aligned}$$

and so forth.

219. Considering particularly the relation last written, I say that it may be broken up into three relations, viz.

$$\begin{aligned} g_{(1^3)} &= G^3_{(1)} - 3\,G_{(1^2)}\,G_{(1)} + 3\,G_{(1^3)}, \\ g_{(21)} &= G_{(2)}\,G_{(1)} - G_{(21)}, \\ g_{(3)} &= G_{(3)}, \end{aligned}$$

for it is easy to see that the two sides of the unfractured operator relation must produce upon any operand the same result identically. Hence after operation those functions which are separations of the same function must separately vanish, and hence we must have the equivalences of operations above set forth.

220. In general there exists the relation

$$\frac{(-)^{\Sigma\pi-1}(\Sigma\pi-1)!}{\pi_1!\,\pi_2!\ldots\ldots}\,g_{(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)} = \sum_j \frac{(-)^{\Sigma j-1}(\Sigma j-1)!}{j_1!\,j_2!\ldots\ldots}\,G^{j_1}_{(J_1)}\,G^{j_2}_{(J_2)}\ldots\ldots,$$

the summation being for all separations

$$(J_1)^{j_1}(J_2)^{j_2}\ldots\ldots$$

of the partition $(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)$.

221. This result gives the general relation between the partition g and the partition G operators, and should be compared with the formula which expresses a function symbolized by a single part in terms of separations of $(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)$.

222. The relation when reversed is

$$\begin{aligned} &(-)^{\Sigma\pi-1}G_{(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)} \\ &\quad = \sum_j \frac{(-)^{\Sigma j-1}(\Sigma\pi_1-1)!\;(\Sigma\pi_2-1)!\ldots\ldots}{j_1!\,j_2!\ldots\ldots\;\pi_{11}!\,\pi_{12}!\ldots\ldots\;\pi_{21}!\,\pi_{22}!\ldots\ldots}\,g^{j_1}_{(p_{11}^{\pi_{11}}p_{12}^{\pi_{12}}\ldots\ldots)}\,g^{j_2}_{(p_{21}^{\pi_{21}}p_{22}^{\pi_{22}}\ldots\ldots)}\ldots\ldots, \end{aligned}$$

the summation being for every separation of the partition $(p_1^{\pi_1}p_2^{\pi_2}\ldots\ldots)$.

223. The following results of operations should be remarked:

$$G_{(p^{\pi}_1{}_1 p_2^{\pi_2}\ldots\ldots)}\,s_{p_1}^{\pi_1}s_{p_2}^{\pi_2}\ldots\ldots = 1,$$

$$\frac{1}{j_1!}\,\frac{1}{j_2!}\ldots\ldots g^{j_1}_{(p_{11}^{\pi_{11}}p_{12}^{\pi_{12}}\ldots\ldots)}g^{j_2}_{(p_{21}^{\pi_{21}}p_{22}^{\pi_{22}}\ldots\ldots)}\ldots\ldots(p_{11}^{\pi_{11}}p_{12}^{\pi_{12}}\ldots\ldots)^{j_1}(p_{21}^{\pi_{21}}p_{22}^{\pi_{22}}\ldots\ldots)^{j_2}\ldots\ldots = 1.$$

§19.

224. There is a more extensive law of reciprocity than that established in §10 of the Third Memoir. The latter sprang from three identities of the form

$$\begin{aligned} 1+A_0+A_1x \quad &+A_2x^2 \quad +\ldots = e^n(1+\alpha_1x)(1+\alpha_2x)\ldots\left(1+\frac{1}{\alpha_1x}\right)\left(1+\frac{1}{\alpha_2x}\right)\ldots, \\ &+A_{-1}\frac{1}{x} + A_{-2}\frac{1}{x^2} + \ldots \end{aligned}$$

which for brevity may be written

$$F(A)=f(\alpha).$$

225. We may instead consider any number of such identities; but first of all, for the sake of simplicity, let us consider four identities

$$\begin{aligned} F(A)&=f(\alpha),\\ F(B)&=f(\beta),\\ F(C)&=f(\gamma),\\ F(D)&=f(\delta),\end{aligned}$$

and add the auxiliary identity

$$F(K)=f(\varkappa).$$

226. Assume the quantities herein involved to be connected by the two relations

$$\begin{array}{l}1+K_0+K_1y\ +K_2y^2\ +\ldots \\ \quad +K_{-1}\dfrac{1}{y}+K_{-2}\dfrac{1}{y^2}+\ldots\end{array} = \prod_s \left(\begin{array}{l}1+B_0+\alpha_sB_1y\ +\alpha_s^2B_2y^2\ +\ldots \\ \quad +\dfrac{1}{\alpha_s}B_{-1}\dfrac{1}{y}+\dfrac{1}{\alpha_s^2}B_{-2}\dfrac{1}{y^2}+\ldots\end{array}\right),$$

$$\begin{array}{l}1+D_0+D_1y\ +D_2y^2\ +\ldots \\ \quad +D_{-1}\dfrac{1}{y}+D_{-2}\dfrac{1}{y^2}+\ldots\end{array} = \prod_s \left(\begin{array}{l}1+K_0+\gamma_sK_1y\ +\gamma_s^2K_2y^2\ +\ldots \\ \quad +\dfrac{1}{\gamma_s}K_{-1}\dfrac{1}{y}+\dfrac{1}{\gamma_s^2}K_{-2}\dfrac{1}{y^2}+\ldots\end{array}\right).$$

Denoting $\Sigma\alpha_1^m$ by $(m)_\alpha$, it has been shown in §10 that these relations lead to the identities

$$\begin{aligned}(m)_\varkappa &= (m)_\alpha(m)_\beta,\\ (m)_\delta &= (m)_\varkappa(m)_\gamma,\end{aligned}$$

so that eliminating $(m)_\varkappa$ we have

$$(m)_\delta = (m)_\alpha(m)_\beta(m)_\gamma;$$

this is equivalent to the result of the elimination of the quantities K between the two assumed relations; in fact we find immediately

$$\begin{array}{l}1+D_0+D_1y\ +D_2y^2\ +\ldots \\ \quad +D_{-1}\dfrac{1}{y}+D_{-2}\dfrac{1}{y^2}+\ldots\end{array} = \prod_s\prod_t \left(\begin{array}{l}1+B_0+\alpha_s\gamma_tB_1y\ +\alpha_s^2\gamma_t^2B_2y^2\ +\ldots \\ \quad +\dfrac{1}{\alpha_s\gamma_t}B_{-1}\dfrac{1}{y}+\dfrac{1}{\alpha_s^2\gamma_t^2}B_{-2}\dfrac{1}{y^2}+\ldots\end{array}\right),$$

and the result is then reached by taking logarithms and expanding in powers of y.

In the found relation

$$(m)_\delta = (m)_\alpha (m)_\beta (m)_\gamma,$$

m may be any integer, positive, zero or negative.

227. The expression $(m)_\delta$ remains unchanged for any permutation of the sets of quantities

$$\begin{array}{l} \alpha_1, \alpha_2, \alpha_3, \ldots . \\ \beta_1, \beta_2, \beta_3, \ldots . \\ \gamma_1, \gamma_2, \gamma_3, \ldots . \end{array}$$

and hence every symmetric function of the quantities

$$\delta_1, \delta_2, \delta_3, \ldots .$$

remains unchanged for all permutations amongst the three sets mentioned.

228. Thus if we have found

$$(s_1^{\sigma_1} s_2^{\sigma_2} s_3^{\sigma_3} \ldots .)_\delta = \ldots . + J(p_1^{\pi_1} p_2^{\pi_2} \ldots .)_\alpha (\lambda_1^{l_1} \lambda_2^{l_2} \ldots .)_\beta (\mu_1^{m_1} \mu_2^{m_2} \ldots .)_\gamma + \ldots .,$$

we necessarily have

$$(s_1^{\sigma_1} s_2^{\sigma_2} \ldots .)_\delta = \ldots . + J\{(p_1^{\pi_1} p_2^{\pi_2} \ldots .)_\alpha (\lambda_1^{l_1} \lambda_2^{l_2} \ldots .)_\beta (\mu_2^{m_1} \mu_1^{m_2} \ldots .)_\gamma + 5 \text{ similar expressions obtained by permuting } \alpha, \beta \text{ and } \gamma\} + \ldots .$$

229. The two assumed relations lead, as shown in §10, to the operator relations

$$\begin{aligned} {}_\beta g_m &= (m)_\alpha \, {}_\kappa g_m, \\ {}_\kappa g_m &= (m)_\gamma \, {}_\delta g_m, \end{aligned}$$

and thence

$$(m)_\delta \, {}_\delta g_m = (m)_\alpha \, {}_\alpha g_m = (m)_\beta \, {}_\beta g_m = (m)_\gamma \, {}_\gamma g_m,$$

showing the invariant character of the operation

$$(m)_\delta \, {}_\delta g_m,$$

for any transformation of a function of the quantities D into a function of either of the sets of quantities A, B, C as given by the relations assumed.

230. Since

$${}_\beta g_m = (m)_\alpha (m)_\gamma \, {}_\delta g_m,$$

we may write

$$\begin{aligned} & {}_\beta g_0 + {}_\beta g_1 y - \tfrac{1}{2} {}_\beta g_2 y^2 + \ldots . \\ & \quad + {}_\beta g_{-1} \frac{1}{y} - \tfrac{1}{2} {}_\beta g_{-2} \frac{1}{y^2} + \ldots . \\ & = (0)_\alpha (0)_\gamma \, {}_\delta g_0 + (1)_\alpha (1)_\gamma \, {}_\delta g_1 y - \tfrac{1}{2} (2)_\alpha (2)_\gamma \, {}_\delta g_2 y^2 + \ldots . \\ & \quad + (\bar{1})_\alpha (\bar{1})_\gamma \, {}_\delta g_{-1} \frac{1}{y} - \tfrac{1}{2} (\bar{2})_\alpha (\bar{2})_\gamma \, {}_\delta g_{-2} \frac{1}{y^2} + \ldots ., \end{aligned}$$

and by taking the exponential of each side, we reach by previous work the relation

$$\begin{aligned}1 + {}_\beta G_0 + {}_\beta G_1 y \quad + {}_\beta G_2 y^2 \quad + \dots\\ + {}_\beta G_{-1}\frac{1}{y} + {}_\beta G_{-2}\frac{1}{y^2} + \dots\\ = \prod_s\prod_t \Big(1 + {}_\delta G_0 + \alpha_s\gamma_t\,{}_\delta G_1 y \quad + \alpha_s^2\gamma_t^2\,{}_\delta G_2 y^2 \quad + \dots\\ + \frac{1}{\alpha_s\gamma_t}\,{}_\delta G_{-1}\frac{1}{y} + \frac{1}{\alpha_s^2\gamma_t^2}\,{}_\delta G_{-2}\frac{1}{y^2} + \dots\Big)\end{aligned}$$

showing that in any relation, connecting the quantities D with the quantities B, we obtain an operator relation by writing

$${}_\beta G_m \text{ for } D_m,$$

and

$${}_\delta G_m \text{ for } B_m.$$

231. In the relation just established we may make any permutation of the letters α, β, γ, so that, regarding the quantities D as expressed in terms of the quantities A or C, we may write

$${}_\alpha G_m \text{ for } D_m,$$
$${}_\delta G_m \text{ for } A_m,$$

or

$${}_\gamma G_m \text{ for } D_m,$$
$${}_\delta G_m \text{ for } C_m$$

in either case.

232. Consider now the quantities D expressed in terms of the quantities C, so that by multiplication we obtain a relation such as

$$D_{p_1}^{\pi_1}D_{p_2}^{\pi_2}\dots = \dots + L(\lambda_1^{l_1}\lambda_2^{l_2}\dots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\dots)_\beta C_{s_1}^{\sigma_1}C_{s_2}^{\sigma_2}\dots + \dots \qquad \text{I}$$

and also two others such as

$$D_{\lambda_1}^{l_1}D_{\lambda_2}^{l_2}\dots = \dots + M(p_1^{\pi_1}p_2^{\pi_2}\dots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\dots)_\beta C_{s_1}^{\sigma_1}C_{s_2}^{\sigma_2}\dots + \dots \qquad \text{II}$$

$$D_{\mu_1}^{m_1}D_{\mu_2}^{m_2}\dots = \dots + N(\lambda_1^{l_1}\lambda_2^{l_2}\dots)_\alpha(p_1^{\pi_1}p_2^{\pi_2}\dots)_\beta C_{s_1}^{\sigma_1}C_{s_2}^{\sigma_2}\dots + \dots \qquad \text{III}$$

233. Assume, moreover,

$$(s_1^{\sigma_1}s_2^{\sigma_2}\dots)_\delta = \dots + J\{(\lambda_1^{l_1}\lambda_2^{l_2}\dots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\dots)_\beta(p_1^{\pi_1}p_2^{\pi_2}\dots)_\gamma + \dots\} + \dots \qquad \text{IV}$$

The relation I yields the operator relation

$${}_\gamma G_{p_1}^{\pi_1}\,{}_\gamma G_{p_2}^{\pi_2}\dots = \dots + L(\lambda_1^{l_1}\lambda_2^{l_2}\dots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\dots)_\beta\,{}_\delta G_{s_1}^{\sigma_1}\,{}_\delta G_{s_2}^{\sigma_2}\dots + \dots,$$

which, performed upon opposite sides of IV, gives

$$\dots + L(\lambda_1^{l_1}\lambda_2^{l_2}\dots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\dots)_\beta = \dots + J(\lambda_1^{l_1}\lambda_2^{l_2}\dots)_\alpha(\mu_1^{m_1}\mu_2^{m_2}\dots)_\beta + \dots,$$

whence $$L = J,$$
and similarly, $$M = N = J;$$
that is, $$L = M = N.$$

234. Hence in the relation

$$D_{p_1}^{\pi_1} D_{p_2}^{\pi_2} \ldots . = \ldots . + L(\lambda_1^{l_1}\lambda_2^{l_2}\ldots .)_\alpha (\mu_1^{m_1}\mu_2^{m_2}\ldots .)_\beta C_{s_1}^{\sigma_1} C_{s_2}^{\sigma_2} \ldots . + \ldots . ,$$

if any permutation be impressed upon the three partitions

$$(\lambda_1^{l_1}\lambda_2^{l_2}\ldots .),\ (\mu_1^{m_1}\mu_2^{m_2}\ldots .),\ (p_1^{\pi_1}p_2^{\pi_2}\ldots .),$$

the numerical coefficient L remains unchanged.

235. More generally consider n identities

$$\begin{aligned} F(A_1) &= f(\alpha_1), \\ F(A_2) &= f(\alpha_2), \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ F(A_n) &= f(\alpha_n), \end{aligned}$$

and therewith, $n - 3$ auxiliary identities

$$\begin{aligned} F(\mathrm{K}_1) &= f(\varkappa_1), \\ F(\mathrm{K}_2) &= f(\varkappa_2), \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ F(\mathrm{K}_{n-3}) &= f(\varkappa_{n-3}). \end{aligned}$$

236. Assume to exist between the quantities involved $n - 2$ relations, viz.

$$\begin{aligned} &1 + \mathrm{K}_{1,0} + \mathrm{K}_{1,1}y + \mathrm{K}_{1,2}y^2 + \ldots \\ &\quad + \mathrm{K}_{1,-1}\frac{1}{y} + \mathrm{K}_{1,-2}\frac{1}{y^2} + \ldots \end{aligned} = \prod_s \left(\begin{aligned} &1 + A_{1,0} + \alpha_{2,s}A_{1,1}y + \alpha_{2,s}^2 A_{1,2}y^2 + \ldots \\ &\quad + \frac{1}{\alpha_{2,s}}A_{1,-1}\frac{1}{y} + \frac{1}{\alpha_{2,s}^2}A_{1,-2}\frac{1}{y^2} + \ldots \end{aligned} \right)$$

which for brevity write

$$\Phi(\mathrm{K}_1) = \phi(\alpha_2, \alpha_1),$$

and also
$$\begin{aligned} \Phi(\mathrm{K}_2) &= \phi(\alpha_3, \varkappa_1), \\ &\ldots\ldots\ldots \\ &\ldots\ldots\ldots \\ \Phi(\mathrm{K}_{n-3}) &= \phi(\alpha_{n-2}, \varkappa_{n-4}), \\ \Phi(A_n) &= \phi(\alpha_{n-1}, \varkappa_{n-3}). \end{aligned}$$

237. Observe that the relation

$$\Phi(\mathrm{K}_s) = \phi(\alpha_{s+1}, \varkappa_{s-1}),$$

is symmetrical with regard to the sets of quantities

$$\alpha_{s+1,1},\ \alpha_{s+1,2},\ \alpha_{s+1,3},\ \ldots\ \text{and}\ \varkappa_{s-1,1},\ \varkappa_{s-1,2},\ \varkappa_{s-1,3},\ \ldots$$

238. We can now eliminate all the auxiliary quantities.

We have

$$\begin{aligned}
(m)_{\kappa_1} &= (m)_{\alpha_2}(m)_{\alpha_1},\\
(m)_{\kappa_2} &= (m)_{\alpha_3}(m)_{\kappa_1},\\
&\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\\
(m)_{\kappa_{n-3}} &= (m)_{\alpha_{n-2}}(m)_{\kappa_{n-4}},\\
(m)_{\alpha_n} &= (m)_{\alpha_{n-1}}(m)_{\kappa_{n-3}},
\end{aligned}$$

and thence

$$(m)_{\alpha_n} = (m)_{\alpha_1}(m)_{\alpha_2}\ldots(m)_{\alpha_{n-1}},$$

and thence, as before, we obtain the identity

$$\begin{aligned}
1 + A_{n,0} + A_{n,1}y &+ A_{n,2}y^2 + \ldots\\
&+ A_{n,-1}\frac{1}{y} + A_{n,-2}\frac{1}{y^2} + \ldots
\end{aligned}$$

$$= \prod_{s_2}\ldots\prod_{s_{n-1}}\left(\begin{array}{l} 1 + A_{1,0} + \alpha_{2,s_2}\alpha_{3,s_3}\ldots\alpha_{n-1,s_{n-1}}A_{1,1}y + \alpha^2_{2,s_2}\alpha^2_{3,s_3}\ldots\alpha^2_{n-1,s_{n-1}}A_{1,2}y^2 + \ldots\\ + \dfrac{1}{\alpha_{2,s_2}\alpha_{3,s_3}\ldots\alpha_{n-1,s_{n-1}}}A_{1,-1}\dfrac{1}{y} + \dfrac{1}{\alpha^2_{2,s_2}\alpha^2_{3,s_3}\ldots\alpha^2_{n-1,s_{n-1}}}A_{1,-2}\dfrac{1}{y^2} + \ldots \end{array}\right)$$

239. This result is invariant for every substitution that may be impressed upon the sets of quantities

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots\ \alpha_{n-1}.$$

240. Proceeding by the same method, as in the preceding particular case, it is easy to prove that if the identity above given lead to a result

$$\begin{aligned}
A^{l_{n,1}}_{n,\lambda_{n,1}}A^{l_{n,2}}_{n,\lambda_{n,2}}\ldots &= \ldots\\
&+ L\,(\lambda_{21}^{l_{21}}\lambda_{22}^{l_{22}}\ldots)_{\alpha_2}(\lambda_{31}^{l_{31}}\lambda_{32}^{l_{32}}\ldots)_{\alpha_3}\ldots(\lambda_{n-1,1}^{l_{n-1,1}}\lambda_{n-1,2}^{l_{n-1,2}}\ldots)_{\alpha_{n-1}}A^{l_{11}}_{1,\lambda_{11}}A^{l_{12}}_{1,\lambda_{12}}\ldots + \ldots,
\end{aligned}$$

wherein L is a numerical coefficient, the number L remains unaltered when any substitution is impressed upon the $n-1$ partitions

$$(\lambda_{s_1}^{l_{s1}}\lambda_{s_2}^{l_{s2}}\ldots);$$
$$s = 2, 3, \ldots n.$$

241. This theorem expresses a very general law of symmetry. For its further examination I shall restrict myself to partitions which contain positive integers, zero excluded. This restriction is made with the sole idea of avoiding complexity in the expressions.

242. Consider these relations

$$1 + A_{s,1}x + A_{s,2}x^2 + \ldots . = (1 + \alpha_{s,1}x)(1 + \alpha_{s,2}x) \ldots . ,$$
$$(s = 1,\ 2, \ldots . n),$$

and

$$1 + K_{s,1}x + K_{s,2}x^2 + \ldots . = (1 + \varkappa_{s,1}x)(1 + \varkappa_{s,2}x) \ldots . ,$$
$$\{s = 1,\ 2, \ldots . (n - 3)\}.$$

243. The relations, assumed to exist between the quantities involved, become

$$1 + K_{1,1}y + K_{1,2}y^2 + \ldots = \prod_{s_2} (1 + \alpha_{2,s_2} A_{1,1}y + \alpha^2_{2,s_2} A_{1,2}y^2 + \ldots),$$
$$1 + K_{2,1}y + K_{2,2}y^2 + \ldots = \prod_{s_3} (1 + \alpha_{3,s_3} K_{1,1}y + \alpha^2_{3,s_3} K_{1,2}y^2 + \ldots),$$
. . . .
$$1 + K_{n-3,1}y + K_{n-3,2}y^2 + \ldots = \prod_{s_{n-2}} (1 + \alpha_{n-2,s_{n-2}} K_{n-4,1}y + \alpha^2_{n-2,s_{n-2}} K_{n-4,2}y^2 + \ldots),$$
$$1 + A_{n,1}y + A_{n,2}y^2 + \ldots = \prod_{s_{n-1}} (1 + \alpha_{n-1,s_{n-1}} K_{n-3,1}y + \alpha^2_{n-1,s_{n-1}} K_{n-3,2}y^2 + \ldots),$$

y being an undetermined quantity.

244. Multiplying out the right-hand sides of these identities, equating coefficients of like powers of y in each and employing the partition notation, we obtain

$$K_{1,1} = (1)_{\alpha_2} A_{1,1},$$
$$K_{1,2} = (2)_{\alpha_2} A_{1,2} + (1^2)_{\alpha_2} A^2_{1,1},$$
$$K_{1,3} = (3)_{\alpha_2} A_{1,3} + (21)_{\alpha_2} A_{1,2} A_{1,1} + (1^3)_{\alpha_2} A^3_{1,1},$$
. . . .
$$K_{2,1} = (1)_{\alpha_3} K_{1,1},$$
$$K_{2,2} = (2)_{\alpha_3} K_{1,2} + (1^2)_{\alpha_3} K^2_{1,1},$$
$$K_{2,3} = (3)_{\alpha_3} K_{1,3} + (21)_{\alpha_3} K_{1,2} K_{1,1} + (1^3)_{\alpha_3} K^3_{1,1},$$
. . . .
. . . .
$$K_{n-3,1} = (1)_{\alpha_{n-2}} K_{n-4,1},$$
$$K_{n-3,2} = (2)_{\alpha_{n-2}} K_{n-4,2} + (1^2)_{\alpha_{n-2}} K^2_{n-4,1},$$
$$K_{n-3,3} = (3)_{\alpha_{n-2}} K_{n-4,3} + (21)_{\alpha_{n-2}} K_{n-4,2} K_{n-4,1} + (1^3)_{\alpha_{n-2}} K^3_{n-4,1},$$
. . . .
$$A_{n,1} = (1)_{\alpha_{n-1}} K_{n-3,1},$$
$$A_{n,2} = (2)_{\alpha_{n-1}} K_{n-3,2} + (1^2)_{\alpha_{n-1}} K^2_{n-3,1},$$
$$A_{n,3} = (3)_{\alpha_{n-1}} K_{n-3,3} + (21)_{\alpha_{n-1}} K_{n-3,2} K_{n-3,1} + (1^3)_{\alpha_{n-1}} K^3_{n-3,1},$$
. . . .

245. We can now, by direct and successive substitution, express the quantities

$$A_{n,s}$$

in terms of the quantities

$$A_{1,s},$$

and the symmetric functions

$$(\;)_{\alpha_2},\ (\;)_{\alpha_3},\ \ldots.\ (\;)_{\alpha_{n-1}},$$

and thence form directly any product

$$A_{n,s_1}A_{n,s_2}A_{n,s_3},\ \ldots.$$

expressed in the same manner.

246. We can also verify the relations

$$A_{n,1}=(1)_{\alpha_2}(1)_{\alpha_3}\ldots.\ (1)_{\alpha_{n-1}}A_{1,1},$$
$$A_{n,1}^2-2A_{n,2}=(2)_{\alpha_2}(2)_{\alpha_3}\ldots.\ (2)_{\alpha_{n-1}}(A_{1,1}^2-2A_{1,2}),$$
$$A_{n,1}^3-3A_{n,2}A_{n,1}+3A_{n,3}=(3)_{\alpha_2}(3)_{\alpha_3}\ldots.\ (3)_{\alpha_{n-1}}(A_{1,1}^3-3A_{1,2}A_{1,1}+3A_{1,3}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(m)_{\alpha_n}=(m)_{\alpha_2}(m)_{\alpha_3}\ldots.\ (m)_{\alpha_{n-1}}(m)_{\alpha_1},$$

which have been previously established.

247. Reverting to the law of symmetry, which it is convenient to express in the form

$$A_{n,\lambda_{n,1}}^{l_{n,1}}A_{n,\lambda_{n,2}}^{l_{n,2}}\ldots=\ldots+L\,(\lambda_{21}^{l_{21}}\lambda_{22}^{l_{22}}\ldots)_{\alpha_2}\ldots(\lambda_{n-1,1}^{l_{n-1,1}}\lambda_{n-2,2}^{l_{n-2,2}}\ldots)_{\alpha_{n-1}}A_{1\lambda_{11}}^{l_{11}}A_{1\lambda_{12}}^{l_{12}}\ldots+\ldots,$$

where L is unchanged for any substitution impressed upon the $n-1$ partitions

$$(\lambda_{s1}^{l_{s1}}\lambda_{s2}^{l_{s2}}\ldots.)$$
$$(s=2,\ 3,\ \ldots.\ n);$$

I recall that in the present instance the partitions are restricted to contain as parts, positive, non-zero integers.

248. Guided by the distribution theorems of the first and second memoirs in this Journal, we may enquire the meaning of the number L in the theory of distributions and of the law of symmetry connected with it which is here brought forward.

§20.

249. In the first and second memoir I considered the distribution of objects of a certain type into parcels of a certain type and obtained a law of symmetry

by observing that it was immaterial whether an object was supposed attached to a parcel or a parcel attached to an object.

250. The consideration of both objects and parcels is in point of fact unnecessary. The parcels may be considered to be objects also, but of a different nature, and the distribution to be of objects of the first set with objects of the second set so as to form a number of pairs of objects, each pair consisting of an object from each set.

251. The result of this distribution may be regarded as the formation of a new set of objects of a two-fold character. Thus if the two sets of objects be

$$a_1,\ a_1,\ a_2,\ a_2, \text{ the first set,}$$
$$b_1,\ b_1,\ b_1,\ b_2, \text{ the second set,}$$

we may make a distribution

$$a_1b_1,\ a_1b_1,\ a_2b_1,\ a_2b_2,$$

and look upon this as a new set of four 2-fold objects. We may then say that we have distributed objects of type (2^2) with objects of type (31) so as to form two-fold objects of type (21^2).*

252. The new method of statement arises naturally from the observed reciprocity between parcels and objects. Together with the new set of two-fold objects we may now consider another set of objects, say

$$c_1,\ c_1,\ c_1,\ c_2,$$

and making any distribution

$$\begin{array}{cccc} a_1b_1, & a_1b_1, & a_2b_1, & a_2b_2, \\ c_1\ , & c_1\ , & c_1\ , & c_2\ , \end{array}$$

we arrive at a new set of three-fold objects

$$a_1b_1c_1,\ a_1b_1c_1,\ a_2b_1c_1,\ a_2b_2c_2,$$

which constitutes objects (3-fold) of type (21^2) obtained by distributing objects (2-fold) of type (21^2) with objects of type (31).

253. Thus from the three sets of objects

$$a_1,\ a_1,\ a_2,\ a_2 \text{ of type } (2^2),$$
$$b_1,\ b_1,\ b_1,\ b_2 \text{ of type } (31),$$
$$c_1,\ c_1,\ c_1,\ c_2 \text{ of type } (31),$$

---

* In the first memoir this was stated to be a distribution of objects of type (2^2) into parcels of type (31) with a partition of restriction (21^2).

we have obtained a distribution

$$a_1b_1c_1,\ a_1b_1c_1,\ a_2b_1c_1,\ a_2b_2c_2,$$

which constitutes objects (3-fold) of type (21^2).

254. Consider the problem of finding the number of distributions obtained from these three sets of objects, given by their types, so that a distribution may be constituted of 3-fold objects of given type (21^2).

255. The two sets of objects

$$\begin{array}{l} a_1,\ a_1,\ a_2,\ a_2, \\ b_1,\ b_1,\ b_1,\ b_2, \end{array}$$

may be distributed into sets of two-fold objects of a variety of types. Each of these sets may be then distributed with the objects

$$c_1,\ c_1,\ c_1,\ c_2,$$

and one or more sets of three-fold objects of type (21^2) may or may not be thus reached.

256. Forming the relations

$$\begin{aligned} A_{3,1} &= (1)_2 \mathrm{K}_{1,1}, \\ A_{3,3} &= (3)_2 \mathrm{K}_{1,3} + (21)_2 \mathrm{K}_{1,2} \mathrm{K}_{1,1} + (1^3)_2 \mathrm{K}_{1,1}^3, \end{aligned}$$

we find

$$A_{3,3} A_{3,1} = \ldots . + 2\,(2^2)_2 \mathrm{K}_{1,2} \mathrm{K}_{1,1}^2 + \ldots . ,$$

which (see first memoir) shows that when objects of type (2^2) are distributed with objects of type (31), the set of two-fold objects formed is necessarily of type (21^2), and they are 2 in number.

257. We have now to find the number of distributions of objects of type (21^2) with objects of type (31) so that the distributions may be of type (21^2).

Writing

$$\begin{aligned} \mathrm{K}_{1,1} &= (1)_1 T_1, \\ \mathrm{K}_{1,2} &= (2)_1 T_2 + (1^2)_1 T_1^2 \end{aligned}$$

we find

$$\mathrm{K}_{1,2} \mathrm{K}_{1,1}^2 = \ldots . + 2\,(31)_1 T_2 T_1^2 + \ldots .$$

From this it appears that any set of objects of type (21^2) may be distributed in two different ways with any set of objects of type (31) in such wise that the distribution is of type (21^2).

258. Combining the two results, we have

$$A_{3,3} A_{3,1} = \ldots . + 4\,(2^2)_2 (31)_1 T_2 T_1^2 + \ldots . ,$$

showing that the whole number of distributions, of the given type, of the three sets of objects is 4.

These are in fact

$$\begin{aligned}&a_1b_1c_1,\ a_1b_1c_1,\ a_2b_1c_1,\ a_2b_2c_2;\\&a_1b_1c_1,\ a_1b_1c_1,\ a_2b_1c_2,\ a_2b_2c_1;\\&a_2b_1c_1,\ a_2b_1c_1,\ a_1b_1c_1,\ a_1b_2c_2;\\&a_2b_1c_1,\ a_2b_1c_1,\ a_1b_1c_2,\ a_1b_2c_1.\end{aligned}$$

259. Consider n sets of objects of types

$$P_1^{(1)},\ P_1^{(2)},\ \ldots\ldots P_1^{(r_1)} \qquad (r_1 = n)$$

and let it be demanded to find the number of distributions into n-fold objects of type

$$T.$$

260. The two sets of objects whose types are $P_1^{(1)}$, $P_1^{(2)}$ may be distributed into a set of two-fold objects of r_2 different types, say

$$P_{12}^{(1)},\ P_{12}^{(2)},\ \ldots\ldots P_{12}^{(r_2)}.$$

261. Selecting at pleasure any one of these sets of two-fold objects, say one of type P_{12}, we may distribute it with the set of objects of type $P_1^{(3)}$ and thus obtain a set of three-fold objects which may be of r_3 different types; say these are

$$P_{123}^{(1)},\ P_{123}^{(2)},\ \ldots\ldots P_{123}^{(r_3)}.$$

262. Again selecting one of these sets at pleasure, we may proceed successively until finally we arrive at a set of n-fold objects which may be of r_n different types which may be denoted by

$$P_{123\ldots}^{(1)}\ ,\ P_{123\ldots n}^{(2)},\ \ldots\ldots P_{123\ldots n}^{(r_n)}.$$

One of these types may or may not be identical with the given type T.

263. Performing the process above indicated in all possible ways, we reach all the distributions into n-fold objects of the given type T.

264. The analytical process for arriving at the number of such distributions is simple and elegant. We have merely to combine the successive processes into a single process, as has been already done in the simple case considered which has gone before.

265. Recalling the previous notation and writing down the relation

$$A_{n,\lambda_{n,1}}^{l_{n,1}} A_{n,\lambda_{n,2}}^{l_{n,2}} \ldots = \ldots + L\,(\lambda_{21}^{l_{21}}\lambda_{22}^{l_{22}}\ldots)_{\alpha_2}\ldots(\lambda_{n-1,1}^{l_{n-1,1}}\lambda_{n-2,2}^{l_{n-2,2}}\ldots)_{\alpha_{n-1}} A_{1\lambda_{11}}^{l_{11}}\, A_{1\lambda_{12}}^{l_{12}}\ldots + \ldots$$

I now state definitely the meaning to be attached to the number L which may be gathered from the preceding.

266. Let there be $n-1$ sets of objects of types

$$\begin{array}{l}(\lambda_{21}^{l_{21}}\lambda_{22}^{l_{22}}\ldots\ldots),\\(\lambda_{31}^{l_{31}}\lambda_{32}^{l_{32}}\ldots\ldots),\\\ldots\ldots\ldots\\\ldots\ldots\ldots\\(\lambda_{n,1}^{l_{n,1}}\lambda_{n,2}^{l_{n,2}}\ldots\ldots).\end{array}$$

The number of distributions into $n-1$-fold sets of objects of type

$$(\lambda_{11}^{l_{11}}\lambda_{12}^{l_{12}}\ldots\ldots)$$

is equal to L.

267. From this statement it is immediately obvious that any substitution whatever may be impressed upon the $n-1$ partitions

$$(\lambda_{s1}^{l_{s1}}\lambda_{s2}^{l_{s2}}\ldots\ldots),$$
$$(s=2,\ 3,\ \ldots\ldots n),$$

and hence in the above written identity the number L is unchanged when any substitution is impressed upon these $n-1$ partitions.

268. This distribution theorem thus involves an intuitive proof of the general law of symmetry.

269. It should be borne in mind that the general identity is constructed through the medium of the identities

$$\begin{array}{l}K_{1,1}=(1)_{a_2}A_{1,1},\\K_{1,2}=(2)_{a_2}A_{1,2}+(1^2)_{a_2}A_{1,1}^2,\\K_{1,3}=(3)_{a_2}A_{1,3}+(21)_{a_2}A_{1,2}A_{1,1}+(1^3)_{a_2}A_{1,1}^3,\\\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}$$

$$s=2.\ 3,\ \ldots,\ (n-3)\left\{\begin{array}{l}K_{s,1}=(1)_{a_s+1}K_{s-1,1},\\K_{s,2}=(2)_{a_s+1}K_{s-1,2}+(1^2)_{a_s+1}K_{s-1,1}^2,\\K_{s,3}=(3)_{a_s+1}K_{s-1,3}+(21)_{a_s+1}K_{s-1,2}K_{s-1,1}+(1^3)_{a_s+1}K_{s-1,1}^3,\\\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right.$$

$$\begin{array}{l}A_{n,1}=(1)_{a_{n-1}}K_{n-3,1},\\A_{n,2}=(2)_{a_{n-1}}K_{n-3,2}+(1^2)_{a_{n-1}}K_{n-3,1}^2,\\A_{n,3}=(3)_{a_{n-1}}K_{n-3,3}+(21)_{a_{n-1}}K_{n-3,2}K_{n-3,1}+(1^3)_{a_{n-1}}K_{n-3,1}^3\\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}$$

270. I propose to obtain the general result for the case of $n=4$ and three objects in each set. There are then three sets of objects.

We have

$$
\begin{aligned}
K_{1,1} &= (1)_{\alpha_2} A_{1,1},\\
K_{1,2} &= (2)_{\alpha_2} A_{1,2} + (1^2)_{\alpha_2} A_{1,1}^2,\\
K_{1,3} &= (3)_{\alpha_2} A_{1,3} + (21)_{\alpha_2} A_{1,2}A_{1,1} + (1^3)_{\alpha_2} A_{1,1}^3,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
A_{4,1} &= (1)_{\alpha_3} K_{1,1},\\
A_{4,2} &= (2)_{\alpha_3} K_{1,2} + (1^2)_{\alpha_3} K_{1,1}^2,\\
A_{4,3} &= (3)_{\alpha_3} K_{1,3} + (21)_{\alpha_3} K_{1,2}K_{1,1} + (1^3)_{\alpha_3} K_{1,1}^3,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

in order to eliminate the quantities K.

271. The result is, writing $(3)_{\alpha_s} = (3)_s$ for brevity, and so on,

$$
\begin{aligned}
A_{4,3} = {} & (3)_3(3)_2 A_{1,3} + \{(3)_3(21)_2 + (21)_3(3)_2\} A_{1,2}A_{1,1} + \{(3)_3(1^3)_2 \\
& + (1^3)_3(3)_2\} A_{1,1}^3 + (21)_3(21)_2 (A_{12}A_{11} + A_{11}^3) \\
& + 3\{(21)_3(1^3)_2 + (1^3)_3(21)_2\} A_{1,1}^3 + 6\,(1^3)_3(1^3)_2 A_{1,1}^3,\\
A_{4,2}A_{4,1} = {} & (3)_3(3)_2 A_{12}A_{11} + \{(3)_3(21)_2 + (21)_3(3)_2\}(A_{12}A_{11} + A_{11}^3) \\
& + 3\{(3)_3(1^3)_2 + (1^3)_3(3)_2\} A_{11}^3 + (21)_3(21)_2 (A_{12}A_{11} + 4A_{11}^3) \\
& + 9\{(21)_3(1^3)_2 + (1^3)_3(21)_2\} A_{11}^3 + 18\,(1^3)_3(1^3)_2 A_{11}^3,\\
A_{4,1}^3 = {} & (3)_3(3)_2 A_{11}^3 + 3\{(3)_3(21)_2 + (21)_3(3)_2\} A_{11}^3 + 6\{(3)_3(1^3)_2 + (1^3)_3(3)_2\} A_{11}^3 \\
& + 9\,(21)_3(21)_2 A_{11}^3 \qquad + 18\{(21)_3(1^3)_2 + (1^3)_3(21)_2\} A_{11}^3 \\
& \qquad\qquad + 36\,(1^3)_3(1^3)_2 A_{11}^3.
\end{aligned}
$$

272. The symmetry is manifest, and representing by

$$[(3)(21)(1^3)]$$

a distribution of three sets of objects of types (3), (21) and (1^3) and so forth, we obtain for the numbers of the different types

$$
\begin{aligned}
[(3)\ ,\ (3)\ ,\ (3)\ ] &= A_{1,3},\\
[(3)\ ,\ (3)\ ,\ (21)] &= A_{1,2}A_{1,1},\\
[(3)\ ,\ (3)\ ,\ (1^3)\ ] &= A_{1,1}^3,\\
[(3)\ ,\ (21),\ (21)] &= A_{1,2}A_{1,1} + A_{1,1}^3,\\
[(3)\ ,\ (21),\ (1^3)\ ] &= 3A_{1,1}^3,\\
[(3)\ ,\ (1^3)\ ,\ (1^3)\ ] &= 6A_{1,1}^3,\\
[(21),\ (21),\ (21)] &= A_{1,2}A_{1,1} + 4A_{1,1}^3,\\
[(21),\ (21),\ (1^3)\ ] &= 9A_{1,1}^3,\\
[(21),\ (1^3)\ ,\ (1^3)\ ] &= 18A_{1,1}^3,\\
[(1^3)\ ,\ (1^3)\ ,\ (1^3)\ ] &= 36A_{1,1}^3,
\end{aligned}
$$

where for example the seventh of these results is to be interpreted as indicating that of three sets of objects of types (21), (21) and (21) respectively, there is one distribution of type (21) and four of type (1^3).

273. Taking the objects to be

$$\begin{aligned} &a_1, a_1, a_2 \text{ of type } (21),\\ &b_1, b_1, b_2 \text{ of type } (21),\\ &c_1, c_1, c_2 \text{ of type } (21), \end{aligned}$$

the five distributions are in fact

$$a_1b_1c_1,\ a_1b_1c_1,\ a_2b_2c_2 \text{ of type } (21)$$

and

$$\left.\begin{matrix} a_1b_1c_1, & a_1b_1c_2, & a_2b_2c_1\\ a_1b_1c_1, & a_1b_2c_2, & a_2b_1c_1\\ a_1b_2c_1, & a_1b_1c_1, & a_2b_1c_2\\ a_1b_1c_2, & a_1b_2c_1, & a_2b_1c_1 \end{matrix}\right\}\text{ of type } (1^3),$$

and observe that there are no others.

274. I have extended the subject of these four memoirs in a paper under the title "Memoir on Symmetric Functions of the Roots of Systems of Equations" in the Philosophical Transactions of the Royal Society of London, Vol. 181 (1890), A, pp. 481–536.

WOOLWICH, *January* 10, 1890.